# Renormalization Effects in a Dilute Bose Gas

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**Abstract**

The low-density expansion for a homogeneous interacting Bose gas at zero temperature can be formulated as an expansion in powers of $\sqrt{\rho a^3}$, where $\rho$ is the number density and $a$ is the S-wave scattering length. Logarithms of $\rho a^3$ appear in the coefficients of the expansion. We show that these logarithms are determined by the renormalization properties of the effective field theory that describes the scattering of atoms at zero density. The leading logarithm is determined by the renormalization of the pointlike $3 \to 3$ scattering amplitude.

The successful achievement of Bose-Einstein condensation of atomic gases in magnetic traps [1] has created an explosion of interest in Bose gases of atoms. While a qualitative description of the condensation can be obtained using mean field methods [2], a more quantitative treatment requires including corrections from fluctuations around the mean field. The relative magnitude of these corrections grows with the number density of the atoms. They will therefore become more important as higher trap densities are achieved.

In order to develop a deeper understanding of the fluctuations, it is worthwhile to go back to the simpler problem of a homogeneous gas of interacting bosons at zero temperature. This problem was studied intensively in the 1950's [3, 4]. A simple review was given by Yang in 1960 [5]. The properties of the system can be calculated as an expansion in powers of $\sqrt{\rho a^3}$, where $\rho$ is the number density of atoms and $a$ is their S-wave scattering length. For example, the expansion for the energy density has the form

$$\mathcal{E} = \frac{2\pi\rho^2 a}{m}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \left[\frac{8}{3}(4\pi - 3\sqrt{3})\log(\rho a^3) + \kappa\right]\rho a^3 + \ldots\right\}, \qquad (1)$$

where we have set $\hbar = 1$. The coefficient of $\sqrt{\rho a^3}$ was first obtained by Lee and Yang for a hard sphere gas [3]. The coefficient of $\rho a^3 \log(\rho a^3)$ was calculated by Wu, by Hugenholtz and Pines, and by Sawada [4]. The correction $\kappa\rho a^3$ is the first term in the expansion that is sensitive to atomic parameters other than the scattering length. We have recently succeeded in calculating the constant $\kappa$ [6].

In this Letter, we use a minimal subtraction renormalization scheme to deduce the general structure of the low-density expansion. We show that logarithms of $\rho a^3$ are related to the renormalization of the effective field theory that describes the scattering of atoms in the vacuum. In particular, the $\log(\rho a^3)$ term in (1) is related to the renormalization of the $3 \to 3$ scattering amplitude. We reproduce the leading logarithms in previous calculations using simple renormalization group methods. Our approach can also be used to determine the logarithms that appear at higher orders in the low-density expansion.

Our starting point is an effective field theory [7] that describes atoms with momenta much lower than their inverse size, which is on the order of the Bohr radius $a_0$. Since the

range of the interaction potential between 2 or more atoms is also on the order of $a_0$, the interactions appear pointlike on the scale of the de Broglie wavelengths of the atoms. The atoms can therefore be described by a field theory with a hamiltonian density that is a local function of the field:

$$\mathcal{H} \;=\; -\frac{1}{2m}\psi^\dagger\nabla^2\psi \;+\; \frac{1}{4}g(\psi^\dagger\psi)^2 \;+\; \frac{1}{36}g_3(\psi^\dagger\psi)^3 \;+\; \ldots, \tag{2}$$

For simplicity, we have assumed that the atoms have spin 0, so that they can be represented by a single complex field $\psi$. The $(\psi^\dagger\psi)^2$ term represents $2 \to 2$ scattering through an S-wave interaction with scattering length $a$ given by $g = 8\pi a/m$, while the term $(\psi^\dagger\psi)^3$ represents $3 \to 3$ scattering. By adding additional terms that are higher order in the derivatives or in the number of fields, one can describe $n \to n$ scattering of atoms in the vacuum with whatever accuracy is desired. In principle, the coefficients of these terms can be calculated from the $n$-body potentials that describe interatomic interactions. In the absence of such calculations, they can be taken as phenomenological parameters.

By treating the interaction terms as perturbations, we can calculate the amplitudes for scattering of atoms with momenta on the order of $p$ as an expansion in powers of $pa_0$. This expansion is complicated by the presence of ultraviolet divergences. For example, the amplitude for the scattering of two atoms with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, including the first perturbative correction, is

$$g\left[1 \;-\; \frac{mg}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2 - (\mathbf{p}_1 + \mathbf{p}_2)\cdot\mathbf{k} + \mathbf{p}_1\cdot\mathbf{p}_2 - i\epsilon}\right]. \tag{3}$$

The integral, which is ultraviolet divergent, can be regularized by imposing a cutoff $|\mathbf{k}| < \Lambda$. The linear divergence can then be cancelled by adding a counterterm proportional to $mg^2\Lambda(\psi^\dagger\psi)^2$ to the effective hamiltonian (2). The resulting expression for the scattering amplitude is rather complicated, as it includes terms that are suppressed by powers of $p_1/\Lambda$ and $p_2/\Lambda$. A simple analytic result is obtained only in the limit $\Lambda \to \infty$.

A power ultraviolet divergence, such as the linear divergence in (3), indicates extreme sensitivity to short-distance atomic physics that is not accurately described by the effective

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hamiltonian (2). A simple momentum cutoff is not an accurate model for the way atomic physics cuts off the momentum integrals. There is an alternative cutoff procedure, called "minimal subtraction", which is no more accurate a model for the cutoff, but provides an equally accurate description of the long-distance physics and has the virtue of simplicity. In minimal subtraction, linear, quadratic, and other power ultraviolet divergences are removed as part of the regularization scheme by subtracting the appropriate power of $k$ from the momentum space integrand. In the case of the amplitude (3), $1/k^2$ is subtracted from the integrand. The justification for this procedure is that the terms that are subtracted are dominated by short distances and can be cancelled by counterterms in the Hamiltonian. In minimal subtraction, logarithmic ultraviolet divergences are treated differently from power divergences. This is reasonable, because logarithmic ultraviolet divergences represent real physical effects, while power ultraviolet divergences are simply artifacts of the regularization procedure. This difference is reflected in the fact that the coefficient of a power divergence $\Lambda^p$ depends on the regularization prescription, while the coefficient of $\log(\Lambda)$ does not. The reason for this is that the logarithm of $\Lambda$ must match onto the logarithm of some physical momentum scale, and therefore its coefficient has a real physical meaning. We regularize logarithmic ultraviolet divergences by imposing a cutoff $|\mathbf{k}| < \Lambda$ on loop integrals. After using renormalization to remove divergences from subdiagrams, we isolate the divergent terms proportional to $\log(\Lambda)$ and then take the limit $\Lambda \to \infty$ in the remainder. The cutoff $\Lambda$ is called the "renormalization scale". With minimal subtraction, all power divergences and those parts of logarithmic divergences that arise from momenta greater than the renormalization scale $\Lambda$ are absorbed into the coupling constants in the effective hamiltonian.

The advantage of minimal subtraction is that it makes it much easier to disentangle the effects of different momentum scales in multiloop diagrams. With a conventional momentum cutoff, a diagram can be an extremely complicated function of the cutoff $\Lambda$, the external momenta, and the momentum scales that can be formed from the parameters in the hamiltonian. With minimal subtraction, the possible dependence of a diagram on the cutoff is

greatly simplified. The dependence can only be polynomial in $\log(\Lambda)$, with the logarithms arising from logarithmically ultraviolet divergent subdiagrams. This makes it much easier to analyze the divergences in a multiloop diagram. The relative simplicity of minimal subtraction is illustrated by the fact that it gives a simple expression for the scattering amplitude (3) that is independent of the renormalization scale $\Lambda$: $g[1+img|\mathbf{p}_1-\mathbf{p}_2|/(16\pi)]$. The renormalized parameter $g$ is independent of $\Lambda$, and satisfies a trivial renormalization group equation: $\Lambda(d/d\Lambda)g = 0$. With a conventional momentum cutoff $\Lambda$, the amplitude (3) is a complicated function of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\Lambda$. For $\Lambda \gg |\mathbf{p}_1|, |\mathbf{p}_2|$, it reduces to $g(\Lambda)[1 - img|\mathbf{p}_1 - \mathbf{p}_2|/(8\pi)]$, where $g(\Lambda)$ is a renormalized coupling constant that satisfies $\Lambda(d/d\Lambda)g(\Lambda) = -mg^2\Lambda/(4\pi^2)$. The running of $g(\Lambda)$ is generated by a power ultraviolet divergence and therefore has no real physical significance. The scale-invariant parameter $g$ defined by minimal subtraction provides an equally accurate description of the long-distance physics.

In the vacuum, the simplest quantity in which logarithmic ultraviolet divergences appear is the $3 \rightarrow 3$ scattering amplitude. There is a tree-level contribution from the $(\psi^\dagger\psi)^3$ term in the effective hamiltonian, but there are also additional contributions that involve successive $2 \rightarrow 2$ scatterings. They include the 2-loop diagrams shown in Figure 1, which involve 4 successive $2 \rightarrow 2$ scatterings. These diagrams are logarithmically ultraviolet divergent. Removing the linear ultraviolet divergence from a subdiagram of the first diagram in Figure 1 by a subtraction in the integrand and then imposing a cutoff $\Lambda$, we find that the logarithmically divergent term is $-3(4\pi - 3\sqrt{3})m^3g^4\log(\Lambda)/(32\pi^3)$.

The renormalization scale $\Lambda$ represents an arbitrary separation between short-distance effects, which are taken into account through the parameters in the effective hamiltonian (2), and long-distance effects, which are calculated using the effective theory. Physical quantities, such as the $3 \rightarrow 3$ scattering amplitude, should therefore be completely independent of $\Lambda$. The explicit $\Lambda$-dependence from the two-loop diagrams must therefore be cancelled by implicit $\Lambda$-dependence from the coefficient $g_3$ in the tree-level contribution. This statement

can be expressed as a "renormalization group equation":

$$\Lambda \frac{d}{d\Lambda} g_3(\Lambda) \;=\; \frac{3}{32\pi^3}(4\pi - 3\sqrt{3})m^3 g^4 \,. \tag{4}$$

It tells us that the parameter $g_3$ is really a "running coupling constant" that increases logarithmically as the momentum scale is increased. The renormalization scale $\Lambda$ can be interpreted as the inverse of the spacial resolution. As this resolution is decreased, we resolve part of the "pointlike" $3 \to 3$ scattering amplitude into the successive $2 \to 2$ scatterings represented by the diagrams in Figure 1. The contributions from the two diagrams have opposite signs and the net effect is that the coupling constant $g_3$ decreases as $\Lambda$ decreases.

If the running coupling constant $g_3$ is determined at the scale $1/a_0$ of atomic structure, it can be calculated at a lower momentum scale $\Lambda$ by solving the renormalization group equation (4):

$$g_3(\Lambda) \;=\; g_3(1/a_0) \;-\; \frac{3}{32\pi^3}(4\pi - 3\sqrt{3})m^3 g^4 \log\left(\frac{1}{\Lambda a_0}\right) \,. \tag{5}$$

Regardless of the sign of $g_3(1/a_0)$, $g_3(\Lambda)$ eventually turns negative for sufficiently small $\Lambda$. In describing the scattering of atoms with momenta on the order of $p$, it is appropriate to choose the renormalization scale $\Lambda$ to be of order $p$. The coefficients in the perturbation expansion can include logarithms of the form $\log(\Lambda/p)$, which are generated by logarithmically divergent subdiagrams. By choosing $\Lambda$ to be of order $p$, such large logarithms are removed from the coefficients and absorbed into the parameters of the effective hamiltonian.

We now consider the energy density $\mathcal{E}$ of the Bose gas with number density $\rho$. In order for the system to have a homogeneous ground state that is stable, or at least metastable, the scattering length $a$ must be positive. Neglecting for the moment the effects of fluctuations, the field $\psi$ develops a vacuum expectation value $\sqrt{\rho}$. The energy density at tree level is

$$\mathcal{E}_0 \;=\; \frac{1}{4}g\rho^2 \;+\; \frac{1}{36}g_3(\Lambda)\rho^3 \;+\; \dots \,. \tag{6}$$

Setting $g = 8\pi a/m$, where $a$ is the S-wave scattering length, the first term above reproduces the leading term in (1). The 1-loop contribution is the sum of the zero-point energies of

the Bogoliubov modes, with power ultraviolet divergences removed by subtractions in the integrand:

$$\mathcal{E}_1 \;=\; \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \epsilon(k) \;-\; \frac{k^2}{2m} \left( 1 + \frac{gm\rho}{k^2} - \frac{g^2m^2\rho^2}{2k^4} \right) \right] \;, \tag{7}$$

where $\epsilon(k) = k\sqrt{k^2 + 2mg\rho}/(2m)$. This integral reproduces the first correction term in (1).

The correction of order $\rho a^3$ in (1) requires the calculation of two-loop diagrams. However the term proportional to $\rho a^3 \log(\rho a^3)$ can be obtained without any further calculation. The reason is that this term is related to the renormalization of the amplitude for $3 \to 3$ scattering in the vacuum. The two-loop diagrams for the energy density contain logarithmic ultraviolet divergences. From the expression for the Bogoliubov energy, we see that the momentum scale associated with the quasiparticles modes is $\sqrt{2mg\rho}$. The logarithmic ultraviolet divergences from the two-loop diagrams will therefore be proportional to $\log(\Lambda/\sqrt{2mg\rho})$. Large logarithms such as this in the coefficients in the perturbation expansion can be avoided by choosing the renormalization scale $\Lambda$ to be on the order of $\sqrt{2mg\rho}$. With this choice of the renormalization scale, all such logarithms are absorbed into the parameters in the effective hamiltonian. Substituting $\Lambda = \sqrt{16\pi a\rho}$ in (6) and using the expression for $g_3(\Lambda)$ in (5), we reproduce the term containing the logarithm in (1). We have determined the constant $\kappa$ under the logarithm by calculating the 2-loop Feynman diagrams for the energy density explicitly. The details of the calculation will be reported elsewhere [6]. As noted previously [4], the $\rho a^3$ term is the first term in the low density expansion (1) that is sensitive to atomic physics parameters other than the scattering length $a$. The only additional parameter that enters at this order is the pointlike $3 \to 3$ scattering amplitude $g_3$.

The renormalization group together with minimal subtraction can also be used to determine the leading logarithms in the low density expansions for other quantities. Corrections to the sound velocity have been calculated by Beliaev [8], including the $\sqrt{\rho a^3}$ term and the $\rho a^3 \log(\rho a^3)$ term. The logarithm can be obtained by calculating the logarithmic ultraviolet divergences in the two-loop corrections to the propagator. Alternatively, it can be obtained trivially using the methods described above. Including the correction from the $(\psi^\dagger \psi)^3$ term

in (2), the sound velocity at tree level is $v^2 = g\rho/(2m) + g_3(\Lambda)\rho^2/3$. Setting $\Lambda = \sqrt{16\pi a\rho}$ and using the expression (5) for $g_3(\Lambda)$, we reproduce the $\rho a^3 \log(\rho a^3)$ correction calculated by Beliaev.

These methods can also be used to determine the coefficients of the logarithms that appear at higher orders in the low-density expansion for the energy density. For example, there is a $(\rho a^3)^{3/2} \log(\rho a^3)$ correction to the energy density which arises from logarithmically divergent two-loop subdiagrams in 3–loop diagrams. This terms can be determined easily by taking into account the $(\psi^\dagger \psi)^3$ term in the Bogoliubov energy: $\epsilon^2(k) = k^2(k^2 + 2mg\rho)/(4m^2) + g_3(\Lambda)\rho^2 k^2/3$. Inserting this into the 1-loop expression (7) for the energy density, expanding to first order in $g_3$, and using the expression (5) for $g_3(\sqrt{16\pi a\rho})$, we obtain the $(\rho a^3)^{3/2} \log(\rho a^3)$ correction.

Thus far we have only considered logarithms in the low density expansion that are related to the renormalization of the $3 \rightarrow 3$ scattering amplitude. At higher orders in the low density expansion, there are also logarithms that are related to the renormalization of other terms in the effective lagrangian, such as the term $g_4(\psi^\dagger \psi)^4$ which describes $4 \rightarrow 4$ scattering through a point-like interaction. The $4 \rightarrow 4$ scattering amplitude includes logarithmically ultraviolet divergent corrections from 4-loop diagrams that involve 7 successive $2 \rightarrow 2$ scatterings and also from 2-loop diagrams that involve three $2 \rightarrow 2$ scatterings and a $3 \rightarrow 3$ scattering. The explicit $\Lambda$–dependence from these loop diagrams must be cancelled by the implicit $\Lambda$–dependence from the pointlike $4 \rightarrow 4$ scattering amplitude $g_4(\Lambda)$. As a consequence, the renormalization group equation analogous to (4) for $(\Lambda d/d\Lambda)g_4$ includes terms on the right side that are proportional to $m^6 g^7$ and $m^3 g^3 g_3$. The solution for $g_4(\Lambda)$ analogous to (5) includes a term proportional to $m^6 g^7 \log^2(\Lambda a_0)$. Choosing $\Lambda = \sqrt{16\pi a\rho}$ to avoid large logarithms from loop diagrams, we find that the term $g_4\rho^4$ in the mean-field expression for the energy density gives rise to a correction to (1) that is proportional to $(\rho a^3)^2 \log^2(\rho a^3)$.

A renormalization group analysis of the dilute Bose gas at nonzero temperature was recently carried out in Ref. [9]. The authors derived renormalization group equations for the

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chemical potential $\mu$ and for the coupling constant $V_0 = 2g$ using a conventional momentum cutoff. If they had used minimal subtraction, their analysis would not have been modified dramatically. Their equations for $d\mu/dl$ would remain unchanged. In their equations for $dV_0/dl$, there would be an additional term $V_0^2 m\Lambda/(2\pi^2)$ on the right side which cancels the leading power of $\Lambda$ in the equation at $T = 0$. The renormalization group trajectories for these two renormalization schemes would differ significantly only near the initial cutoff, where both calculations would be dominated by cutoff artifacts. They would be essentially identical near the critical point for Bose condensation. The logarithmic evolution of the coupling constant $g_3$ was not seen in the analysis of Ref. [9], because they considered renormalization effects from one-loop diagrams only.

In this Letter, we have shown how the structure of the low-density expansion for a Bose gas is determined by the renormalization properties of the effective hamiltonian that describes the scattering of atoms at zero density. The low density expansion for the energy density has the general form

$$\mathcal{E} \;=\; \frac{\rho^2 a}{m} \sum_{n=0}^{\infty} \sum_{l=0}^{l_n} C_{nl} \, (\rho a^3)^{n/2} \log^l(\rho a^3) \; . \tag{8}$$

The maximum power of the logarithm has been determined to be $l_n = 0, 0, 1, 1, 2$ for $n = 0, 1, 2, 3, 4$, respectively. The dimensionless coefficients $C_{nl}$ are polynomials in the generalized coupling constants of higher order terms in the effective hamiltonian (2), with only a finite number of these coupling constants appearing at any given order in $\sqrt{\rho a^3}$. The coupling constant $g_3$ first appears at order $\rho a^3$. Additional coupling constants enter at order $(\rho a^3)^{3/2}$. The general structure in (8) follows automatically from the renormalization group together with the minimal subtraction renormalization scheme. This powerful method should also be useful for analyzing the corrections from fluctuations around the mean field for atomic gases in magnetic traps.

# References

[1] M.H. Anderson et al., Science **269**, 198 (1995); C.C. Bradley et al., Phys. Rev. Lett. **75**, 1687 (1995); K.B. Davis et al., Phys. Rev. Lett. **75**, 3969 (1995).

[2] G. Baym and C.J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

[3] T.D. Lee and C.N. Yang, Phys. Rev. **105**, 1119 (1957); T.D. Lee, K. Huang, and C.N. Yang, Phys. Rev. **106**, 1135 (1957).

[4] T.T. Wu, Phys. Rev. **115**, 1390 (1959); N.M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959); K. Sawada, Phys. Rev. **116**, 1344 (1959).

[5] C.N. Yang, Physica **26**, 549 (1960).

[6] E. Braaten and A. Nieto, Ohio State preprint OHSTPY-HEP-T-96-024 (in preparation).

[7] H. Georgi, Ann. Rev. Nucl. Part. Sci. **43**, 209 (1993).

[8] Beliaev, Soviet Physics JETP **7**, 299 (1958).

[9] M.Bijlsma and H.T.C. Stoof, Utrecht preprint (cond-mat/9605120).

# Figure Caption

Figure 1. Two-loop diagrams that give logarithmically ultraviolet divergent contributions to the $3 \rightarrow 3$ scattering of atoms.
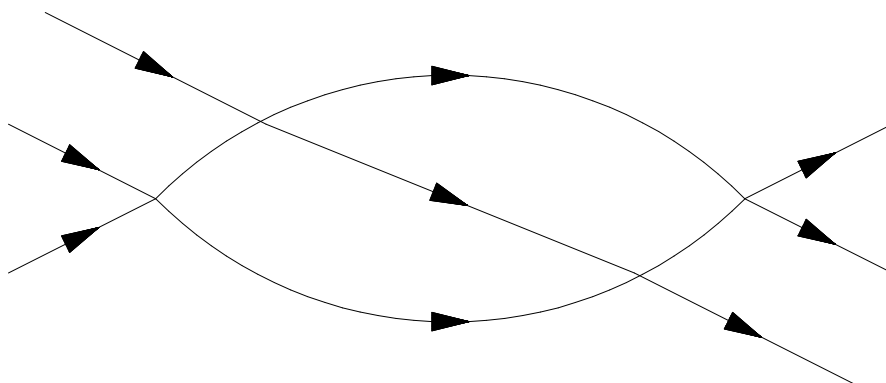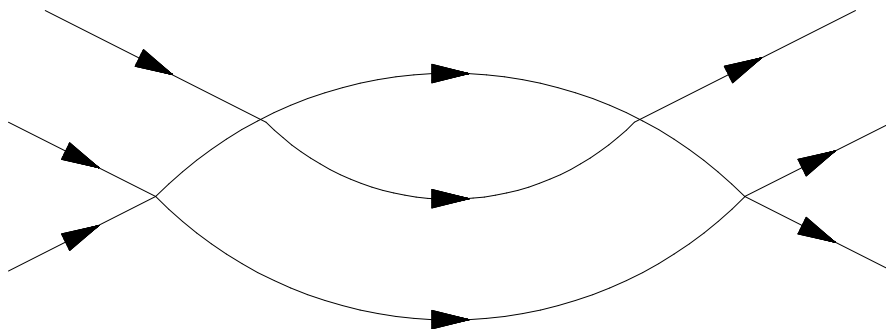
FIGURE 1